

Coarsening Dynamics of Nonequilibrium Chiral Ising Models

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We investigate a nonequilibrium coarsening dynamics of a one-dimensional Ising spin system with chirality. Only spins at domain boundaries are updated so that the model undergoes a coarsening to either of equivalent absorbing states with all spins $+$ or $-$. Chirality is imposed by assigning different transition rates to events at down $(+-)$ kinks and up $(-+)$ kinks. The coarsening is characterized by power-law scalings of the kink density $\rho \sim t^{-\delta}$ and the characteristic length scale $\xi \sim t^{1/z}$ with time t . Surprisingly the scaling exponents vary continuously with model parameters, which is not the case for systems without chirality. These results are obtained from extensive Monte Carlo simulations and spectral analyses of the time evolution operator. Our study uncovers the novel universality class of the coarsening dynamics with chirality.

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Coarsening takes place in various systems such as magnetic systems, binary alloys, and social systems with opinion dynamics. When a system is quenched from a high-temperature disordered phase to a low-temperature ordered phase, a typical size of domains grows in time following a power law

$$\xi \sim t^{1/z}, \quad (1)$$

with dynamic exponent z . It is known that coarsening systems are classified into a few universality classes depending on spatial dimensionality, order parameter symmetry, conservation in dynamics, and so on [1].

The Ising model is one of the best studied coarsening systems. It is symmetric under the global spin inversion (Z_2 symmetry) and has a scalar order parameter. Under the single-spin-flip Glauber dynamics [2] that does not conserve the order parameter, the coarsening dynamics is characterized by $z = 2$. On the other hand, the dynamic exponent is given by $z = 3$ under the spin-exchange Kawasaki dynamics [3] conserving the order parameter. Systems with nonscalar order parameter constitute distinct universality classes [1].

A coarsening process is rather simple in systems with discrete symmetry and nonconserving dynamics in one dimension. Consider a one-dimensional (1D) Ising spin chain with the Glauber dynamics at zero temperature, or equivalently the voter model [4]. In this model, only spins at domain boundaries can flip so that domain walls diffuse and annihilate in pairs. The diffusive nature suggests that the dynamic exponent is given by $z = 2$ and that the domain wall density decays algebraically as

$$\rho \sim t^{-\delta} \quad (2)$$

with an exponent $\delta = 1/2$. These scaling laws are verified by the exact solution [2, 5, 6].

The power-law scaling with $z = 2$ and $\delta = 1/2$ seems to be robust in one dimension. The q -state Potts model

with the zero-temperature Glauber dynamics exhibits the same scaling behavior [5, 7–9]. It is also observed in nonequilibrium systems. Consider the voter model with an additional exchange process of neighboring spins [10]. It is equivalent to the branching annihilating random walk (BAW) model [11, 12], where domain walls diffuse, annihilate in pairs, and branch two offsprings. Despite the branching, the model displays the coarsening with the same exponents [12, 13]. The voter model with a kinetic constraint also displays the same scaling behavior with a logarithmic correction [14].

Most studies on the coarsening have focused on the role of order parameter symmetry [1]. On the other hand, some dynamical systems are characterized by coupled symmetry and little is known about the coarsening dynamics in such systems. In this Letter, we investigate the coarsening dynamics of a 1D Ising spin system which is invariant under the simultaneous inversion of spin and space. Remarkably, the model with the coupled symmetry constitutes a novel universality class that is characterized by continuously varying exponents.

Our study was initially motivated by a flocking phenomenon of self-propelled particles. Flocking means a collective motion of particles, which is easily observed in a group of birds, fish, insects, or animals in nature. Vicsek *et al* [15] proposed a simple model where the motion of each particle i at position \mathbf{r}_i is described by a velocity vector \mathbf{v}_i of a constant speed. Each time step, the direction of \mathbf{v}_i is updated to the average direction of particles within a fixed distance perturbed by a random noise. The system coarsens into a flocking phase when the noise strength is small [15, 16]. Note that spatial isotropy is broken spontaneously due to the motion of particles. Consequently, the model is symmetric under the simultaneous rotation/inversion of the velocity and the space.

In order to investigate the effect of coupled symmetry,

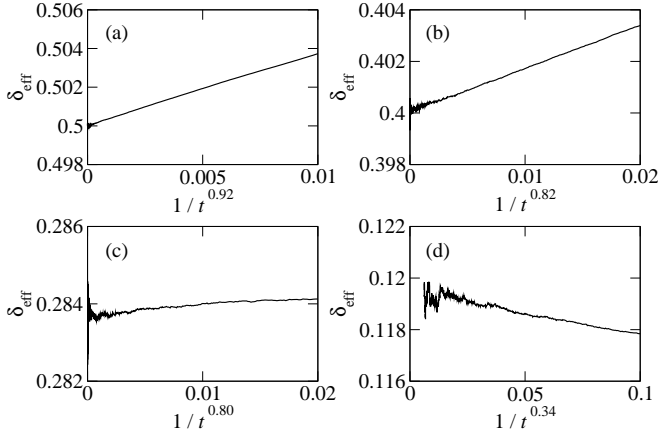


FIG. 1. Plots of effective exponents δ_{eff} for (a) $u = 0$, (b) 0.3, (c) 0.6, and (d) 0.9.

we study an Ising spin system $\{\sigma = (\sigma_1, \dots, \sigma_N)\}$ with $\sigma_i = \pm 1$ in a 1D lattice of N sites under periodic boundary conditions. Only spins near domain walls or kinks are updated following the rule

$$\begin{aligned} &+ - \xrightarrow{u} - +, & - + \xrightarrow{\bar{u}} + -, \\ &+- \xrightarrow{v/2} \begin{cases} ++, \\ --, \end{cases} & -+ \xrightarrow{\bar{v}/2} \begin{cases} ++, \\ --, \end{cases} \end{aligned} \quad (3)$$

where $+$ ($-$) stands for a spin with value $+1$ (-1) and parameters over the arrows denote the transition rates of corresponding events. Without losing generality, we set $u + v = 1$ and $\bar{u} + \bar{v} \leq 1$. The model may be represented with two kinds of domain walls or kinks: A and B corresponding to a *down kink* $+-$ and an *up kink* $-+$, respectively. The spin dynamics are then translated to an unbiased hopping of A (B) with rate v (\bar{v}) and branching of $A \rightarrow ABA$ ($B \rightarrow BAB$) with rate u (\bar{u}). Opposite kinks annihilate in pair upon collision.

Our model is characterized by *chirality*. The transition rates for events associated with down and up kinks are different. This chirality breaks the Z_2 symmetry, but leaves system invariant under the simultaneous inversion of spin and space, $\sigma_i \rightarrow -\sigma_{-i}$. Emphasizing the role of the chirality, the model will be referred to as the nonequilibrium chiral Ising model (NCIM). We remark that chirality is irrelevant for equilibrium Ising systems [17]. However, it turns out to result in an interesting feature in nonequilibrium cases.

The NCIM reduces to the voter model when $v = \bar{v}$ and $u = \bar{u} = 0$, and the asymmetric simple exclusion process (ASEP) [4] when $v = \bar{v} = 0$. A mixture of them was studied in Refs. [18, 19] and was found to display complicated scaling behaviors. When $u = \bar{u}$ and $v = \bar{v}$ (Z_2 symmetric case without chirality), discerning A from B becomes nominal and the model becomes equivalent to the BAW model [12]. It is solvable exactly [13] and the kink density decays with $\delta = 1/2$ for all $u < 1$. Note

TABLE I. Numerical values of δ and z of the MCM for various values of u . For z , we present the results from the eigenspectrum analysis [z (spectrum)] and from the seed simulations [z (seed)]. The numbers in parentheses indicate errors of the last digits.

u	δ	z (spectrum)	z (seed)
0	0.5000(1)	2.0000(6)	1.998(8)
0.1	0.4678(2)	1.8789(5)	1.877(9)
0.2	0.4346(3)	1.7683(6)	1.769(4)
0.3	0.4001(2)	1.6666(6)	1.667(2)
0.4	0.3639(5)	1.5716(8)	1.570(6)
0.5	0.3254(5)	1.4818(7)	1.483(4)
0.6	0.2837(4)	1.3958(4)	1.397(4)
0.7	0.2376(8)	1.3117(8)	1.313(6)
0.8	0.1850(8)	1.227(2)	1.228(3)
0.9	0.1195(3)	1.136(5)	1.139(6)

that the NCIM is different from the so-called directed Ising model in which kinks are biased to a preferred direction [20, 21].

First, we study the maximum chirality case with $\bar{u} = \bar{v} = 0$ where up kinks (B) are immobile and do not branch offsprings. This case is analogous to a 1D version of the flocking model [15] if $+$ ($-$) spin is interpreted as a bird moving to the right (left) and if a bird is assumed to interact with another bird along its moving direction. For convenience, we refer to this case as the maximum chirality model (MCM).

The system is prepared in an anti-ferromagnetic state ($\dots + - + - \dots$) initially. Then, we measure the total kink density and average over N_S samples to obtain $\rho(t)$ for $t \leq 10^7$. Just like the BAW model, we expect that $\rho(t)$ decays in a power-law fashion for all $u < 1$ [22]. So we investigate the behavior of an effective exponent defined as

$$\delta_{\text{eff}}(t) = -\log[\rho(t)/\rho(t/b)]/\log b \quad (4)$$

with a constant b . If the long time behavior of the kink density is given by $\rho(t) = t^{-\delta}(a + ct^{-\zeta})$ up to a leading correction to scaling, the effective exponent in the long time limit behaves as $\delta_{\text{eff}}(t) \simeq \delta + a_1 t^{-\zeta}$, where a_1 is a constant depending on b . So when we draw δ_{eff} against $t^{-\zeta}$ with the correct value of ζ , $\delta_{\text{eff}}(t)$ should approach to δ with a finite slope as $t \rightarrow \infty$.

Figure 1 presents the behavior of the effective exponents for $u = 0, 0.3, 0.6$, and 0.9 . The number of runs for all u is $N_S = 5000$ and system sizes for $u = 0, 0.3, 0.6$, and 0.9 are $N = 2^{24}, 2^{23}, 2^{22}$, and 2^{21} , respectively. These system sizes are large enough that all runs did not fall into absorbing states where all spins have the same sign.

The correction-to-scaling exponent ζ are roughly estimated from a fitting of δ_{eff} to the form $\delta + a_1 t^{-\zeta}$ with

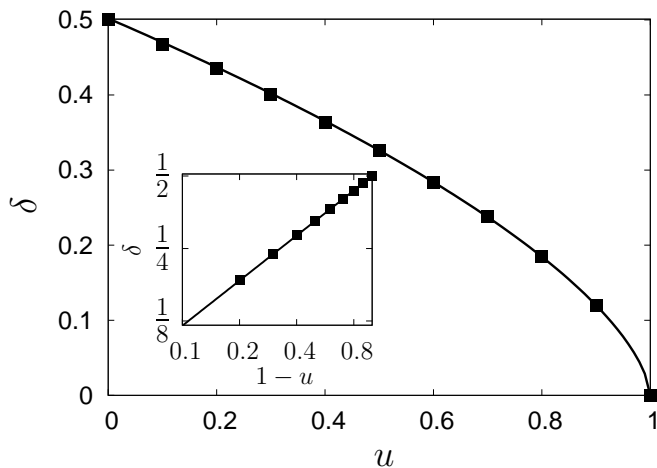


FIG. 2. Plots of δ vs u (symbols) and its fitting function $a_2(1-u)^\chi$ with $a_2 \approx 0.5$ and $\chi \approx 0.62$ (lines). Inset: Log-log plot of δ against $1-u$ together with the fitting function.

three fitting parameters δ , a_1 , and ζ . This procedure can be error-prone and the accuracy of the estimated ζ may be questionable. Still, however, if we draw δ_{eff} against $t^{-\zeta}$ with ζ from a fitting, we observed that δ_{eff} 's intersect the y -axis with finite slope, which allows for estimating δ . We observed that the corrections to scaling become stronger as u approaches 1, which is thought to be a signature of the crossover from the mean-field voter dynamics to the MCM; see below. The analysis of the effective exponent for other values of u is presented in Supplemental Material [22]. Table I summarizes numerical results of δ for various u 's and Fig. 2 illustrates δ against u .

The intriguing feature of the MCM is that the exponent δ varies with u . Furthermore, as u approaches 1, δ seems to show a non-trivial power-law behavior. Indeed, if we fit δ for the region $u \geq 0.6$ using $\delta \approx a_2(1-u)^\chi$, with two fitting parameters a_2 and χ , we found that it fits the data quite well with $a_2 \approx 0.5$ and $\chi \approx 0.62$; see Fig. 2. This singular behavior of δ at $u = 1$ can be understood as a result of a crossover from the mean-field voter dynamics to the MCM.

When u is very close to 1 but not exactly 1, the voter dynamics which occurs with rate $v = 1-u$ happens after many attempts of the ASEP dynamics. Since the stationary state of the ASEP is totally uncorrelated [4], the voter dynamics can happen only after all spins are distributed almost randomly. Hence, if we rescale the time as $\tau = (1-u)t$ and take a limit $u \rightarrow 1$ with τ kept finite, the MCM should be the same as a mean-field voter model on a complete graph. Since coarsening does not occur on a complete graph of infinite size, δ should be zero in the above mentioned limit. Thus, there should be a crossover from the mean-field voter dynamics to the 1D MCM at $u = 1$. Note that the above argument does not depend on whether v and/or \bar{v} are zero or not once \bar{v} approaches zero as $u \rightarrow 1$. Although the accuracy of χ

may be questionable, we can still insist that the crossover is described by the (non-trivial) exponent χ .

We substantiate the Monte Carlo results by studying the spectrum of the time evolution operator of the MCM. In general, a master equation can be mapped to an imaginary time Schrödinger equation with *Hamiltonian* H whose eigenvalues contain most of relevant information of the system. For instance, the directed percolation system has been studied successfully with the eigenspectrum analysis [23–25].

For the MCM, the Hamiltonian takes the form

$$H = \frac{1}{4} \sum_{i=1}^N (1 + \hat{\sigma}_i^z)(1 - \hat{\sigma}_{i+1}^z) - u \sum_{i=1}^N \hat{\sigma}_i^- \hat{\sigma}_{i+1}^+ - \frac{(1-u)}{4} \sum_{i=1}^N \{ \hat{\sigma}_i^- (1 - \hat{\sigma}_{i+1}^z) + (1 + \hat{\sigma}_i^z) \hat{\sigma}_{i+1}^+ \}, \quad (5)$$

where $\hat{\sigma}_i$ is the Pauli spin operator acting on a spin at site i . We label eigenvalues of H as E_n with $n = 1, \dots, 2^N$ and call E_n the energy of the n -th level. Since H is not Hermitian, E_n may have a complex value. The eigenvalues are sorted in the ascending order of $\Re[E_n]$, the real part of E_n . There are two trivial levels with $E_1 = E_2 = 0$ corresponding to the two absorbing states with all spins having the same sign. Other low-lying energy levels with $n > 2$ define the relaxation time as $\tau_n = 1/(\Re[E_n])$.

We have diagonalized numerically the Hamiltonian up to $N = 20$ to obtain the longest relaxation time τ_3 . Since $\tau_3 \sim N^z$, the dynamic exponent z is estimated by extrapolating an effective exponent $z_{\text{eff}}(N) \equiv \ln[\tau_3(N)/\tau_3(N-2)]/\ln[N/(N-2)]$ with the Bulirsch-Stoer (BST) algorithm [26, 27]. The results are summarized in Table I. As an illustration, detailed analysis of z_{eff} for $u = 0.8$ along with the extrapolation according to the BST algorithm is presented in Supplemental Material [22].

We have also performed independent Monte Carlo simulations starting with a single down kink A in the middle of an infinite lattice, which are generally referred to as seed simulations. We have measured the particle spreading distance $\xi(t) \sim t^{1/z}$ to obtain the dynamic exponent z following the similar effective exponent analysis as done for δ . Both results for z are in perfect agreement with each other and vary continuously with u . See Supplemental Material [22] for the full analyses of z .

We have shown that the coarsening dynamics of the MCM is characterized by continuously varying critical exponents. Note that chirality lies both in the ASEP events ($u \neq \bar{u}$) and the voter-model events ($v \neq \bar{v}$). In order to investigate which one is the essential ingredient, we studied two more cases: One is the case with $u = \bar{u}$ and $v \neq \bar{v} = 0$ which will be called the symmetric exclusion and chiral voter (SECV) model and the other is the case with $u \neq \bar{u} = 0$ and $v = \bar{v}$ to be called the chiral exclusion and symmetric voter (CESV) model. The CESV model is a particular limiting case of the model studied

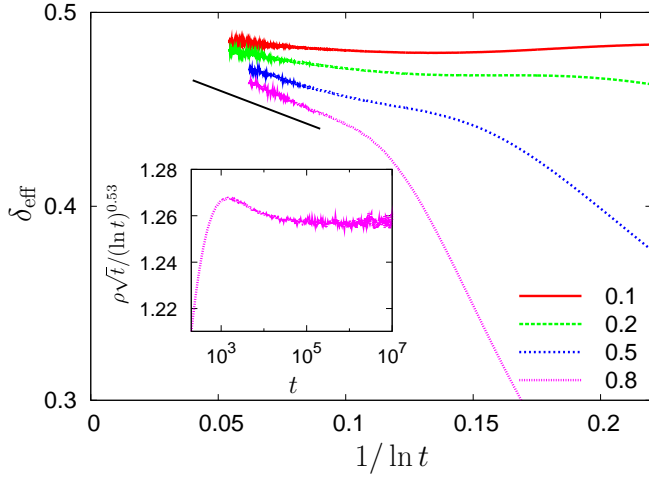


FIG. 3. (Color online) Plots of δ_{eff} as functions of $1/\ln t$ for $u = 0.1, 0.2, 0.5$, and 0.8 (from top to bottom) for the SEC model. A line segment with slope -0.5 is for a guide to the eyes. Inset: Plot of $\rho(t)\sqrt{t}/(\ln t)^{0.53}$ vs t for the SEC model with $u = 0.8$ on a semi-logarithmic scale.

in Ref. [18, 19]. Remind that v is always set to $1 - u$.

The analyses of δ_{eff} for the SEC model with $u = 0.1, 0.2, 0.5$, and 0.8 are summarized in Fig. 3. It seems that $\delta_{\text{eff}}(t)$ for all u approaches $1/2$ with logarithmic corrections. Notice that if there exists a logarithmic correction as

$$\rho(t) \sim (\ln t + C)^\kappa / t^\delta \quad (6)$$

with a constant C , the effective exponent defined in Eq. (4) should behave as $\delta_{\text{eff}}(t) \approx \delta - \kappa/\ln t$ in the asymptotic regime. Thus, if we plot $\delta_{\text{eff}}(t)$ as a function of $1/\ln t$, the effective exponent should intersect the y -axis with slope $-\kappa$. This phenomenon is quite pronounced for the cases of $u = 0.5$ and 0.8 and the slope seems to be around 0.5 . Indeed, if $\rho(t)\sqrt{t}/(\ln t)^{0.53}$ is plotted against t on a semi-logarithmic scale (see Inset of Fig. 3 for the case of $u = 0.8$), a flat region is observable in the long time limit for more than two log-decades. Although the accurate value of κ is hard to estimate, we can conclude that there exist a systematic logarithmic correction as shown in Eq. (6) with the leading scaling exponent $\delta = 1/2$ unchanged in the SEC model.

A logarithmic correction in a 1D coarsening has been reported in a different model [14] that corresponds to the voter model with a weak kinetic constraint. In that model, the kink density decays faster than $1/\sqrt{t}$ as $1/(\sqrt{t} \ln t)$, but in our case it decays slower than $1/\sqrt{t}$. Qualitatively, this slowing down should be attributed to the presence of branching dynamics which increases the number of kinks. However, a quantitative analysis requires further investigation, which is beyond the scope of this Letter. For our purpose, it is enough to conclude that the continuously varying exponents are not due to the chiral voter dynamics.

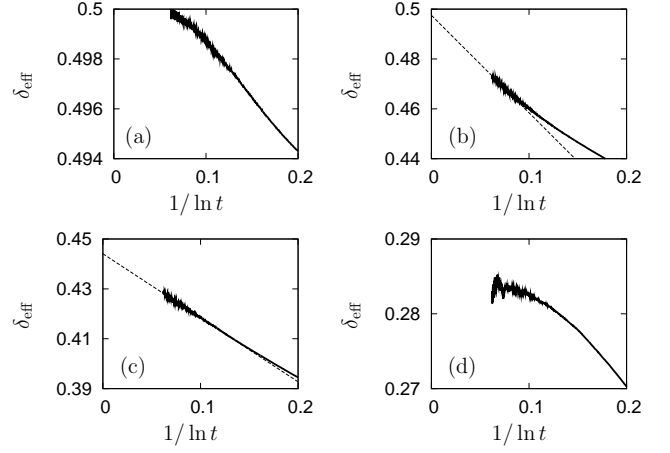


FIG. 4. Plots of δ_{eff} as functions of $1/\ln(t)$ for $u =$ (a) 0.2 , (b) 0.5 , (c) 0.6 , and (d) 0.8 for the CES model. For the case of $u = 0.5$ and 0.6 [(b) and (c)], fitting results of a function $\delta + d/\ln t$ with δ and d fitting parameters are also depicted.

The CES model shows a more intriguing feature. We present the effective exponent data for the density decay in Fig. 4. For $u = 0.2$ [Fig. 4(a)], δ_{eff} seems to approach 0.5 with negligible logarithmic correction. For $u = 0.5$ [Fig. 4(b)], we cannot make a firm conclusion whether $\delta < 0.5$ or $\delta = 0.5$ due to strong correction-to-scaling behavior. Quite interestingly, when $u > 0.5$ [Fig. 4(c) and (d)], δ deviates from 0.5 significantly even under the assumption of a logarithmic correction. So we conclude that the CES model has continuously varying exponents with possible logarithmic corrections when $u > 0.5$. This study shows that the chirality in the spin exchange is responsible for the continuously varying exponents. Nevertheless, it remains open why and when there appears a logarithmic correction in the coarsening process.

To summarize, we have studied the one-dimensional coarsening dynamics of nonequilibrium Ising spin systems with chirality. Although chirality is irrelevant in equilibrium Ising systems, it turns out that the chirality can lead to continuously varying scaling exponents in the nonequilibrium chiral Ising model. In particular, it turns out that the chirality in spin exchange plays a crucial role.

It is rare to observe continuously varying exponents from systems without quenched disorder. The q -state Potts model with zero-temperature Glauber dynamics was studied in Refs. [8, 9]. It was found that the critical exponent describing the power-law decay of the persistent probability varies continuously with q . Nevertheless, the coarsening dynamics is still pure diffusive and characterized by $z = 2$ and $\delta = 1/2$ at all values of q . We notice that continuously varying exponents were reported in a 1D sandpile model without dissipation [28] and that there is actually a parallelism between this sand-

pile model and the MCM. This connection will be discussed elsewhere [29].

Some of Ising spin systems are exactly solvable in one dimension [2, 5, 8, 24], for equations governing the time evolution of correlation functions are closed. In the presence of the chirality, however, the equations are not closed, which makes the exact solution for the NCIM not available in general. Our numerical finding of the universality class with continuously varying exponents can be established more firmly if one find a minimal continuum equation obeying the proper symmetry property. We leave it as a future work [29].

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